

THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH3070 (Second Term, 2017–2018)
Introduction to Topology
Exercise 9b Compact Hausdorff

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let (X, \mathfrak{T}) be a Hausdorff space such that every proper subset of X is compact. Show that the topology is discrete. Do you think the converse is true?
2. Let (X, \mathfrak{T}) be a compact Hausdorff space. Prove that for every $x \in X$ and every open set U containing x , there is a compact set K such that $x \in \text{Int}(K) \subset K \subset U$ (i.e. compact neighborhoods form a local base). As a consequence, the conclusion is true also for locally compact Hausdorff space.

Give an example of a compact non-Hausdorff space such that there is a point which has no local base formed by compact neighborhoods.

3. Let Y be compact Hausdorff. For a mapping $f: X \rightarrow Y$, define

$$G = \{ (x, f(x)) \in X \times Y : x \in X \} .$$

Prove that f is continuous if and only if G is a closed subset of $X \times Y$.

4. It is known that a continuous bijection $f: X \rightarrow Y$ from a compact space X to a Hausdorff space Y is a homeomorphism. Give examples to show that the compactness of X and Hausdorff property of Y cannot be dropped.
5. Show that an infinite set X with cofinite topology is T_1 but not T_2 . What if the set is finite?
6. Let $f, g: X \rightarrow Y$ be continuous mappings. What is the natural requirement for Y if you need one of the following:
 - (a) The set $\{ x \in X : f(x) = y_0 \}$ is closed for every $y_0 \in Y$.
 - (b) The set $\{ x \in X : f(x) = g(x) \}$ is closed.
7. A metric d is defined on $n \times n$ matrices by $d(A, B) = [\text{tr}((A - B)(A - B)^T)]^{1/2}$. Convince yourself that the orthogonal matrices $O(n)$ is compact but $SL(n)$ is not, where

$$\begin{aligned} Q \in O(n) & \text{ iff } QQ^T = \text{identity}, \\ A \in SL(n) & \text{ iff } \det(A) = 1. \end{aligned}$$

8. Show that the one-point compactification of \mathbb{R}^n is homeomorphic to the standard \mathbb{S}^n .
9. Let (X, \mathfrak{T}) be a Hausdorff space. Show that the following two statements are equivalent.
- Each $x \in X$ has a compact neighborhood N containing x .
 - For each $x \in X$ and each neighborhood U of x , there is a compact set $K \subset U \subset X$ such that $x \in \text{Int}(K)$. That is, compact neighborhoods form a local base at each $x \in X$.